Optics & MatLab

William Torruellas
William.torruellas@jhuapl.edu
(W) 1-443-778 9065
(C) 1-443-756 7678

MatLab notions you should be getting familiar

- Matrix and vector manipulation: critical for code vectorization
- Plotting and 2D and 3D graphing
- Min and other Matlab data manipulation functions
- Functions: create your own function
- Functions of functions: fmin, calling functions from a master mfile
- Some GUI design capability

- Today:
  - We will learn about integration in MatLab
  - We will learn how to use matrices to model Gaussian beam propagation

- The E&M Wave Equation, Refraction and Loss/Gain in optical media
- Ray-Optics: representing the propagation of the normal of planar wave fronts. Does not take into account the amplitude of the wave, in other words the propagation of energy. First order analysis of an optical system.
- Today we will look at another particular solution of Maxwell's equation which represents the propagation of well behaved laser beams, both amplitude and phase propagation are well represented by Gaussian beams.

Today's Agenda

- 2nd Homework Lab is due today!

- Gaussian Beams
  - *Properties of Gaussian Beams*

- Modes of Resonant Cavities
  - *Stability Criteria*

- Gaussian Beams in Linear Systems

- <u>References:</u>

1) "Lasers", Tony Siegman, Universtiy Sciences Book 1986
   Chapters: 16&17, 19&20
2) "Optical Electronics", Ammon Yariv, CBS college Publishing
3) Kogelnik and Li,"Laser Beams and Resonators", IEEE proceedings, 54,1312-1329, 1966

Optical Systems applications of Gaussian Beams

Power Projection
(Aperture, Focal Length, Distortion)

Receiving Optics
FoV, FoR, Distortion

Optical Source (W, ϕ, λ, Ap)	Power Projection	**Propagation Material (*Loss, Gain*, Distortion)**	Receiving Optics	*Receiver (Gain, Noise)*
1) Diode laser *2) LED* ***3) Fiber Laser*** ***4) Bulk Laser*** ***5) Black Body***	**1) Fiber coupler** **2) Telescope** **3) Lens**	**1) Vacuum** **2) Atmosphere** **3) Lens/mirror** **4) Materials** **5) Fiber** **6) Amplifier**	***1) Fiber coupler*** **2) Telescope** **3) Lens** *4) Adaptive optics* *5) Pin camera*	*1) Eye* *2) Thermometer* *3) Photo-Diode* *4) PMT* *5) Interferometer* *6) Signal/Image Processing*

Gaussian Beams

Gaussian Beams

Analysis of an approximate solution of Maxwell's equations that relates to laser beams inside and outside the resonator.

Let's start with the wave equation for the electric field in vacuum:

$$\nabla^2 E(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}E(\mathbf{r},t) = 0$$

In analyzing laser beams, we focus on monochromatic electric fields at a single frequency ω, and the solutions of the wave equation we pursue have the form:

$$E(\mathbf{r},t) = \mathcal{E}(\mathbf{r})e^{j\omega t}$$

The wave equation applied to this expression results in the *Helmholtz equation* for $\mathcal{E}(r)$:

$$\nabla^2 \mathcal{E}(\mathbf{r}) + k^2 \mathcal{E}(\mathbf{r}) = 0 \qquad k = \omega / c = 2\pi / \lambda \quad \text{wavenumber}$$

Two possible solutions of the Helmholtz equation are particularly important:

$$\mathcal{E}(\mathbf{r}) = \mathcal{E}_0 e^{-j\mathbf{k}\bullet\mathbf{r}} \qquad\qquad \mathcal{E}(\mathbf{r}) = \frac{A}{r}e^{-jkr}$$

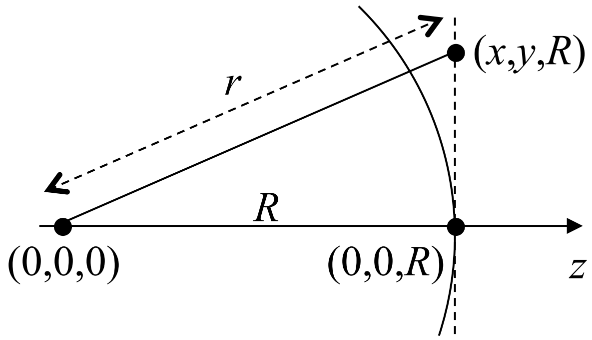
Plane waves **Spherical wave** $(r \neq 0)$

$\mathcal{E}_0$: amplitude (constant) point source at origin; intensity $|\mathcal{E}|^2 \sim 1/r^2$

A: amplitude (constant)

Gaussian Beams

Spherical wave solution $\mathcal{E}(\mathbf{r}) = \frac{A}{r} e^{-jkr}$ $\qquad r = (x^2 + y^2 + R^2)^{1/2} = R\left(1 + \frac{x^2 + y^2}{R^2}\right)^{1/2}$



If the analysis is restricted to a small region around the point $(0,0,R)$ then x^2+y^2 is small compared to R^2 (*paraxial wave*), and in the Taylor series expansion

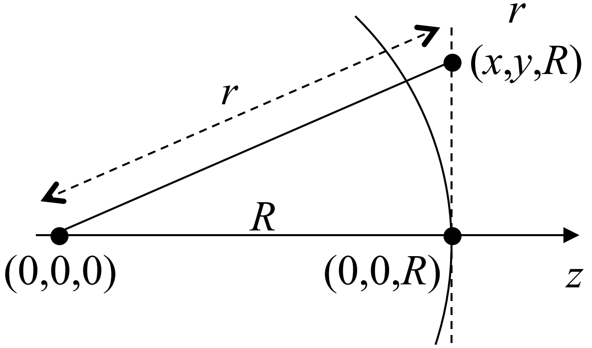
$$\left(1 + \frac{x^2 + y^2}{R^2}\right)^{1/2} = 1 + \frac{x^2 + y^2}{2R^2} - \frac{(x^2 + y^2)^2}{8R^4} + \ldots$$

we can use only the first two terms.

We get $r \approx R + \frac{(x^2 + y^2)}{2R}$ and the product kr can be approximated with $kr \approx kR + \frac{k(x^2 + y^2)}{2R}$

Gaussian Beams

$$\mathcal{E}(\mathbf{r}) = \frac{A}{r} e^{-jkr}$$



$$\left(1 + \frac{x^2 + y^2}{R^2}\right)^{1/2} = 1 + \frac{x^2 + y^2}{2R^2} - \frac{(x^2 + y^2)^2}{8R^4} + \ldots$$

The term $\dfrac{x^2 + y^2}{2R}$ is small compared to R, but not compared to $\lambda = 2\pi/k$.

The field on the plane $z = R$, around $x = 0$, and $y = 0$, becomes $\mathcal{E}(\mathbf{r}) = \dfrac{A}{R} e^{-jkR} e^{-jk(x^2 + y^2)/2R}$

This approximation is frequently used in physical optics. For this approximation to be a good one, the third term in the Taylor expansion of r must be small compared to the wavelength:

$$R\frac{(x^2 + y^2)^2}{8R^4} << \lambda \qquad \frac{(x^2 + y^2)^2}{8\lambda R^3} << 1 \qquad \frac{a^2}{\lambda R} << \left(\frac{R}{a}\right)^2 \quad a = \sqrt{x^2 + y^2}$$

Gaussian Beams

The plane wave and spherical wave solutions of the Helmholtz equation provide insight but cannot represent laser beams. We have to find a beam solution to:

$$\nabla^2 \mathcal{E}(\mathbf{r}) + k^2 \mathcal{E}(\mathbf{r}) = 0$$

In a Gaussian beam, at any plane normal to the propagation direction z, the electric field amplitude is highest on the z axis, and decreases away from it. Therefore we use a form that has a spatially varying amplitude:

$$\mathcal{E}(\mathbf{r}) = \mathcal{E}_0(\mathbf{r}) e^{-jkz}$$

Applying the Helmholtz equation to this field, we get: $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathcal{E}_0(\mathbf{r}) e^{-jkz} + k^2 \mathcal{E}_0(\mathbf{r}) e^{-jkz} = 0$

We can assume that the field amplitude $\mathcal{E}_0(r)$ and its derivative $\partial \mathcal{E}_0(r)/\partial z$ do not vary significantly within a distance of the order of a wavelength, in the z direction:

$$\lambda \left|\frac{\partial \mathcal{E}_0}{\partial z}\right| << |\mathcal{E}_0| \quad \text{and} \quad \lambda \left|\frac{\partial^2 \mathcal{E}_0}{\partial z^2}\right| << \left|\frac{\partial \mathcal{E}_0}{\partial z}\right|$$

and because

$$k = 2\pi/\lambda, \qquad \left|\frac{\partial \mathcal{E}_0}{\partial z}\right| << k|\mathcal{E}_0| \quad \text{and} \quad \left|\frac{\partial^2 \mathcal{E}_0}{\partial z^2}\right| << k\left|\frac{\partial \mathcal{E}_0}{\partial z}\right|$$

Gaussian Beams

The second derivative with respect to z is:

$$\frac{\partial^2}{\partial z^2}\mathcal{E}_0(\mathbf{r})e^{-jkz} = \left(\frac{\partial^2 \mathcal{E}_0}{\partial z^2} - 2jk\frac{\partial \mathcal{E}_0}{\partial z} - k^2\mathcal{E}_0\right)e^{-jkz}$$

but since $\left|\frac{\partial^2 \mathcal{E}_0}{\partial z^2}\right| << k\left|\frac{\partial \mathcal{E}_0}{\partial z}\right|$

$$\frac{\partial^2}{\partial z^2}\mathcal{E}_0(\mathbf{r})e^{-jkz} \approx \left(-2jk\frac{\partial \mathcal{E}_0}{\partial z} - k^2\mathcal{E}_0\right)e^{-jkz}$$

The Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathcal{E}_0(\mathbf{r})e^{-jkz} - 2jk\frac{\partial \mathcal{E}_0(\mathbf{r})}{\partial z}e^{-jkz} - k^2\mathcal{E}_0(\mathbf{r})e^{-jkz} + k^2\mathcal{E}_0(\mathbf{r})e^{-jkz} \approx 0$$

becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2jk\frac{\partial}{\partial z}\right)\mathcal{E}_0(\mathbf{r}) \approx 0$$

(paraxial wave equation) $\nabla_T^2\mathcal{E}_0(\mathbf{r}) - 2jk\frac{\partial \mathcal{E}_0(\mathbf{r})}{\partial z} = 0$

where $\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian operator.

Gaussian Beams

In a plane normal to the direction of propagation z, the intensity of a Gaussian beam can be represented with:

$$I(x,y,z) \sim \left|\mathcal{E}_0(\mathbf{r})\right|^2 e^{-2(x^2+y^2)/w^2(z)}$$

At a transverse distance w from the z axis, the intensity drops by a factor of e^2 (7.389) compared to the z axis value (maximum). The radius of the laser beam spot size is $w(z)$. Guided by the Gaussian formulation, we attempt to find a solution for the paraxial wave equation

$$\nabla_T^2 \mathcal{E}_0(\mathbf{r}) - 2jk\frac{\partial \mathcal{E}_0(\mathbf{r})}{\partial z} = 0$$

a solution of the form

$$\boxed{\mathcal{E}_0(\mathbf{r}) = Ae^{-jk(x^2+y^2)/2q(z)}e^{-jp(z)}}$$

where A is a constant and $q(z)$ and $p(z)$ should be determined to satisfy the paraxial wave equation. If we set

$$\boxed{\frac{1}{q} = \frac{-2j}{kw^2(z)} = \frac{-j\lambda}{\pi w^2(z)}}$$

then the solution gets a Gaussian intensity profile. By setting a q value that depends on z, we enable the spot size to vary with distance, as observed in laser beams.

Next we apply the paraxial wave equation to the attempted solution, to start the derivation of $q(z)$ and $p(z)$.

Gaussian Beams

The transverse Laplacian:

$$\nabla_T^2 \mathcal{E}_0(\mathbf{r}) = A\left[\frac{-2jk}{q} - \frac{k^2}{q^2}(x^2+y^2)\right] e^{-jk(x^2+y^2)/2q(z)} e^{-jp(z)}$$

The first derivative with respect to z:

$$\frac{\partial \mathcal{E}_0(\mathbf{r})}{\partial z} = jA\left[\frac{k}{2}(x^2+y^2)\frac{1}{q^2}\frac{dq}{dz} - \frac{dp}{dz}\right] e^{-jk(x^2+y^2)/2q(z)} e^{-jp(z)}$$

The paraxial wave equation:

$$\nabla_T^2 \mathcal{E}_0(\mathbf{r}) - 2jk\frac{\partial \mathcal{E}_0(\mathbf{r})}{\partial z} = A\left[\frac{k^2}{q^2}(x^2+y^2)\left(\frac{dq}{dz}-1\right) - 2k\left(\frac{dp}{dz}+\frac{j}{q}\right)\right] e^{-jk(x^2+y^2)/2q(z)} e^{-jp(z)} = 0$$

For this equation to hold we need:

$$\frac{dq}{dz} = 1 \qquad \frac{dp}{dz} = -\frac{j}{q}$$

$$q(z) = q_0 + z \qquad p(z) = -j\ln\frac{q_0+z}{q_0}$$

$$q_0 = q(0) \qquad p(0) = 0$$

To account for the most general case, we expect $q(z)$ to be complex and have the form:

$$\boxed{\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{j\lambda}{\pi w^2(z)}}$$

$R(z)$ and $w(z)$ are real functions

Gaussian Beams

$$e^{-jp(z)} = e^{-\ln\frac{q_0+z}{q_0}} = \frac{q_0}{q_0+z} = \frac{1}{1+z/q_0} = \frac{1}{1+z/R_0 - j\lambda z/\pi w_0^2}$$

where $R_0 = R(0)$ and $w_0 = w(0)$ at $z = 0$

Where is $z = 0$? It is an arbitrary choice. Let's choose $z = 0$ to be the plane at which $R = \infty$. Then, $R_0 = \infty$ and

$$\frac{1}{q_0} = \frac{1}{R_0} - \frac{j\lambda}{\pi w_0^2} = -\frac{j\lambda}{\pi w_0^2}$$

We also know that $q(z) = q_0 + z$ which can be written as $\frac{1}{q(z)} = \frac{1}{q_0+z} = \frac{1/q_0}{1+z(1/q_0)}$

Substituting the value of $1/q_0$ we derived above, we get:

$$\frac{1}{q(z)} = \frac{-j\lambda/\pi w_0^2}{1-jz\lambda/\pi w_0^2}$$

and multiplying the denominator and numerator with the conjugate of the denominator, we obtain

$$\frac{1}{q(z)} = \frac{-j\lambda/\pi w_0^2 + z(\lambda/\pi w_0^2)^2}{1+(z\lambda/\pi w_0^2)^2}$$ which should be equivalent to $$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{j\lambda}{\pi w^2(z)}$$

These lead to expressions for $R(z)$ and $w(z)$, by equating separately the real and imaginary parts.

Gaussian Beams

The real part is $\dfrac{z(\lambda/\pi w_0^2)^2}{1+(z\lambda/\pi w_0^2)^2} = \dfrac{1}{R(z)}$ or $R(z) = \dfrac{1}{z(\lambda/\pi w_0^2)^2} + \dfrac{(z\lambda/\pi w_0^2)^2}{z(\lambda/\pi w_0^2)^2}$

$R(z) = z + \dfrac{(\pi w_0^2)^2}{z\lambda^2}$ or $\boxed{R(z) = z + \dfrac{z_0^{\,2}}{z}}$ with $\boxed{z_0 = \dfrac{\pi w_0^2}{\lambda}}$ $(Rayleigh\ \ range)$

The imaginary part is

$$\frac{-j\lambda/\pi w_0^2}{1+(z\lambda/\pi w_0^2)^2} = -\frac{j\lambda}{\pi w^2(z)}$$

or

$$\frac{\pi w^2(z)}{\pi w_0^2} = 1 + (z\lambda/\pi w_0^2)^2$$

leading to

$$\boxed{w(z) = w_0\sqrt{1 + z^2/z_0^2}}$$

(M²)²

The Rayleigh Range defines the length of collimation !!

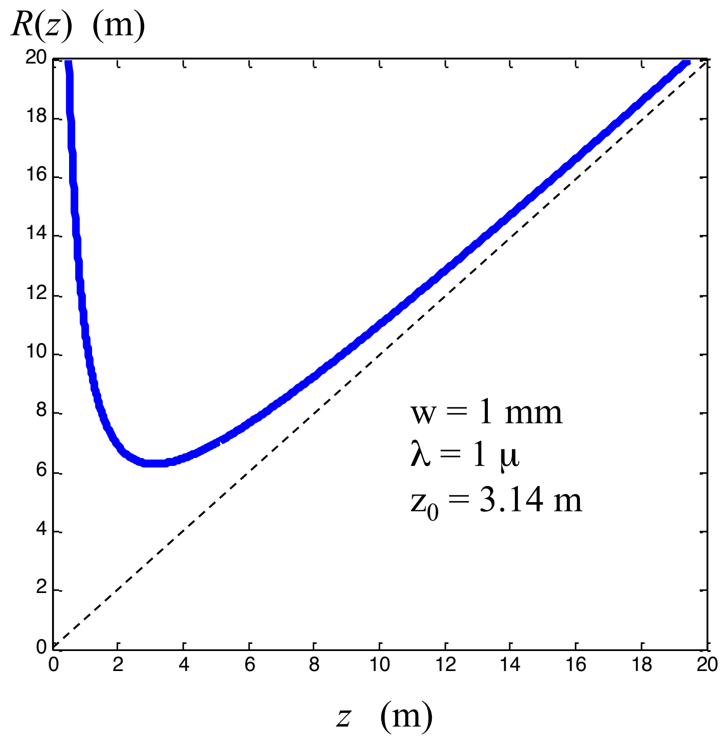
Gaussian Beams

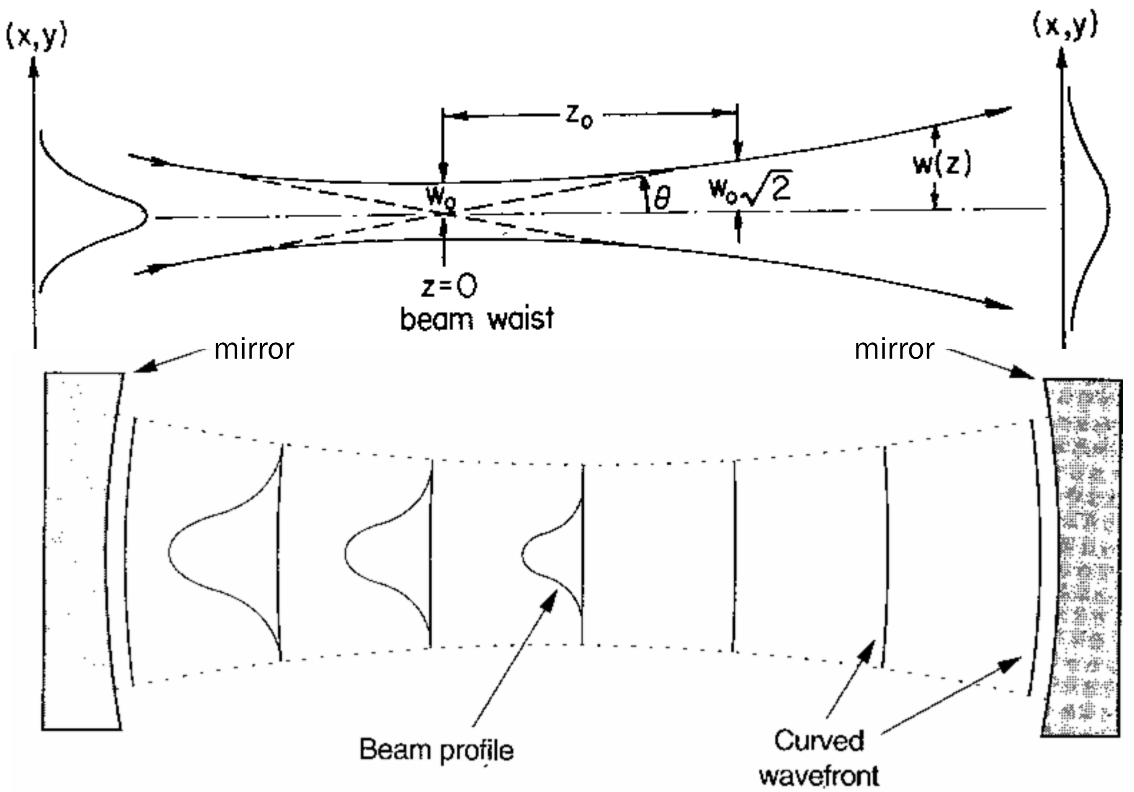
The full expression of the amplitude is

$$\boxed{\mathcal{E}_0(\mathbf{r}) = \frac{Ae^{j\phi(z)}}{\sqrt{1+z^2/z_0^2}}e^{-jk(x^2+y^2)/2R(z)}e^{-(x^2+y^2)/w^2(z)}}$$

with

$$\boxed{R(z) = z + \frac{{z_0}^2}{z} \qquad w(z) = w_0\sqrt{1+z^2/z_0^2} \qquad z_0 = \frac{\pi w_0^2}{\lambda}}$$





Gaussian Beams

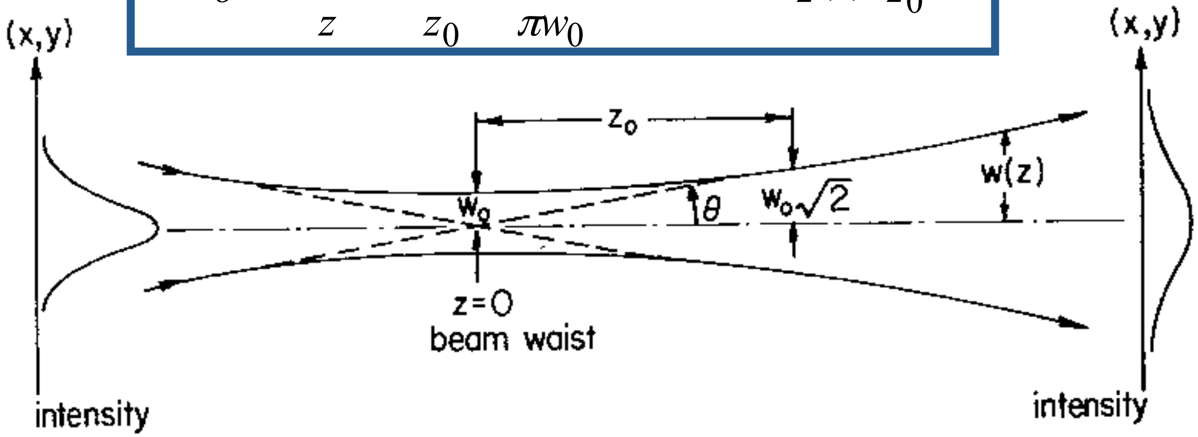
The spot size $w(z)$ is minimal at the plane $z = 0$, where its value is w_0 (beam waist). At the Rayleigh range z_0, the spot size is

$$w(z_0) = w_0\sqrt{2}$$

The Rayleigh range is considered to be a measure of the length of the waist region. A small beam waist produces a short waist region, and a rapid growth in spot size.

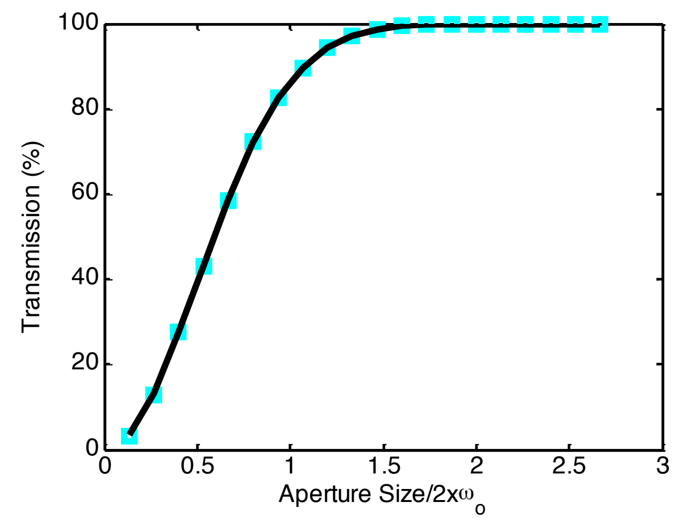
The divergence angle of a Gaussian beam is defined as

$$\boxed{\theta \approx \frac{w(z)}{z} \approx \frac{w_0}{z_0} = \frac{\lambda}{\pi w_0} \qquad z >> z_0}$$

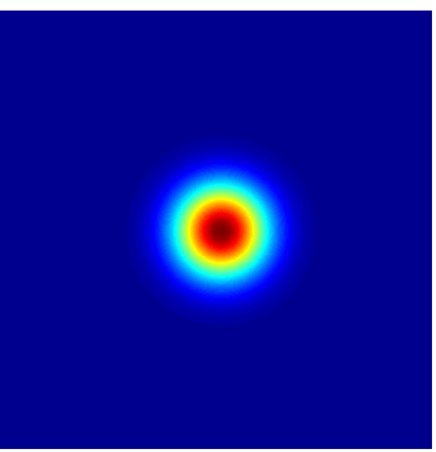


Transmission through an Aperture of a Gaussian Beam

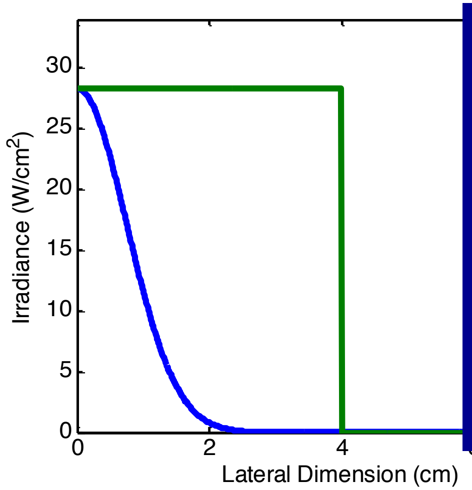
$$T = \int_0^{R\max} 2\pi r\,dr\,\frac{I(r)}{I_o} = \int_0^{R\max} 2\pi r\,dr\;e^{-2\frac{r^2}{\omega_o^2}} = 1 - e^{-2 r_{\max}^2/\omega_o^2}$$



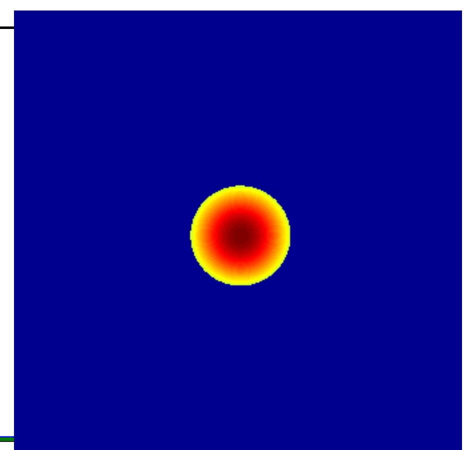
Input Gaussian Field



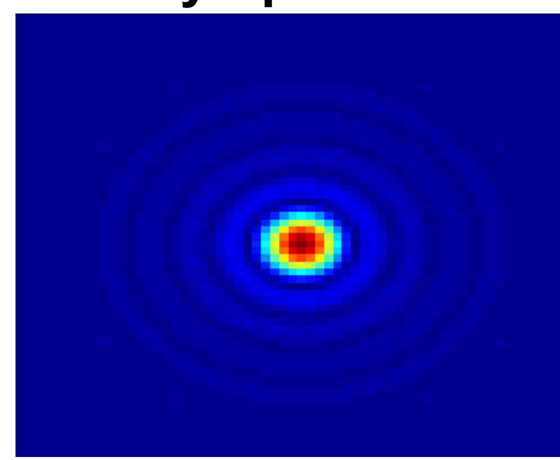
Aperture



Transmitted Gaussian Field



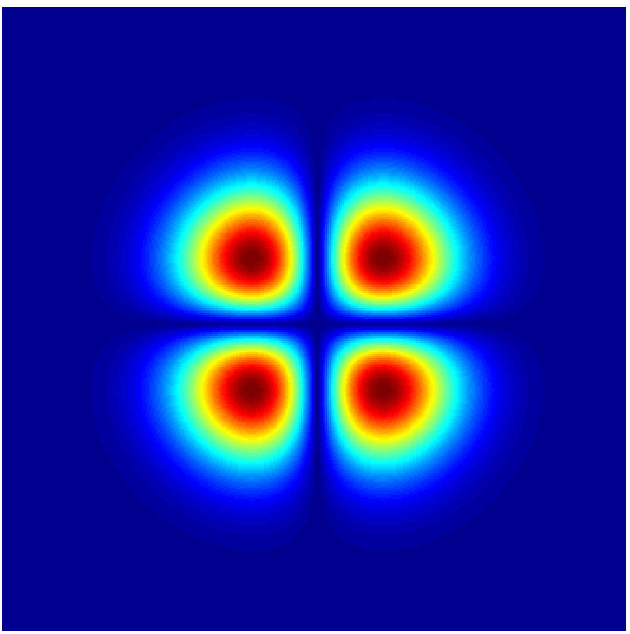
Diffracted Field By Aperture



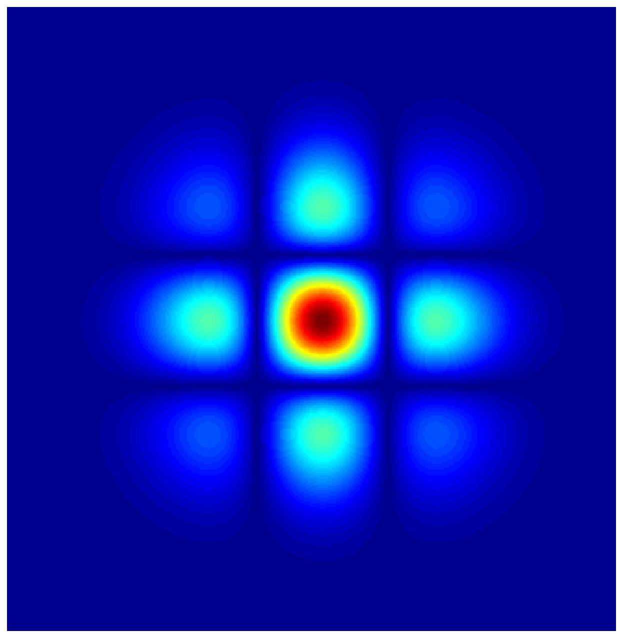
High order Gaussian Modes

$$E_{mn}(r) = \sqrt{\frac{1}{2^{m+n}\, n \;\; m}}\,\frac{1}{\omega(z)}\,H_m(x)H_n(y)e^{i\frac{k}{2}\left(\frac{x^2+y^2}{q(z)}\right)}e^{ik.z}e^{-i(m+n+1)\psi(z)}$$

A similar expression is used in cylindrical coordinates using Laguerre polynomials instead of Hermitte polynomials in Cartesian coordinates



TEM_{11}



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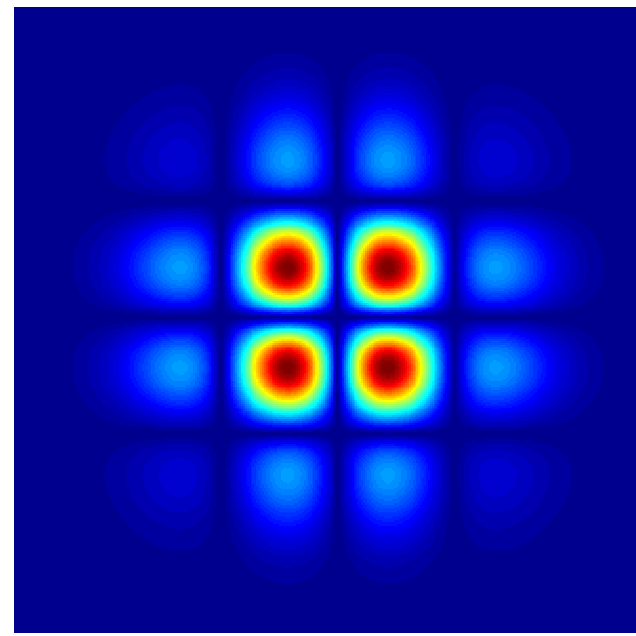


TEM_{33}

Gaussian m.file

- GaussAper.m

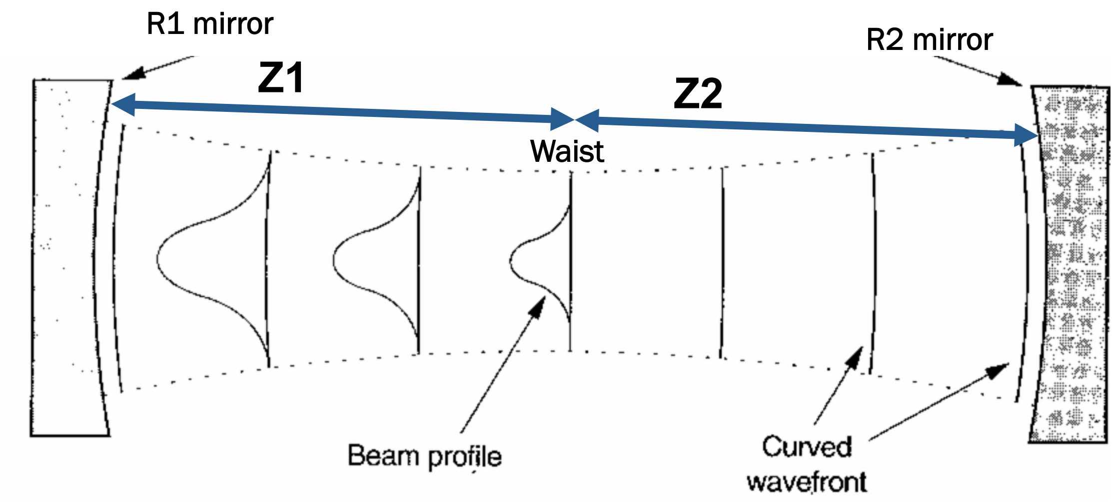
Gaussian Beams
Modes of Laser Cavities

Modes of Resonant Cavity

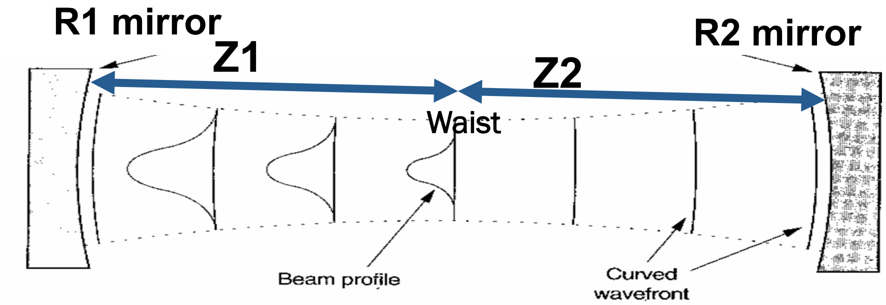
- Phase fronts need to match the boundary conditions provided by the mirror curvatures.
- You also need to find the position of the waist, infinite curvature phase front (plane-wave).

$$R_1 = -z_1 - \left(\frac{\pi\omega_o^2}{\lambda}\right)^2 \frac{1}{z_1} \quad ; \quad R_2 = +z_2 + \left(\frac{\pi\omega_o^2}{\lambda}\right)^2 \frac{1}{z_2}$$

$$z_2 - z_1 = L$$



Modes of Resonant Cavity



$$g_1 = 1 - \frac{L}{R_1} \quad ; \quad g_2 = 1 - \frac{L}{R_2}$$

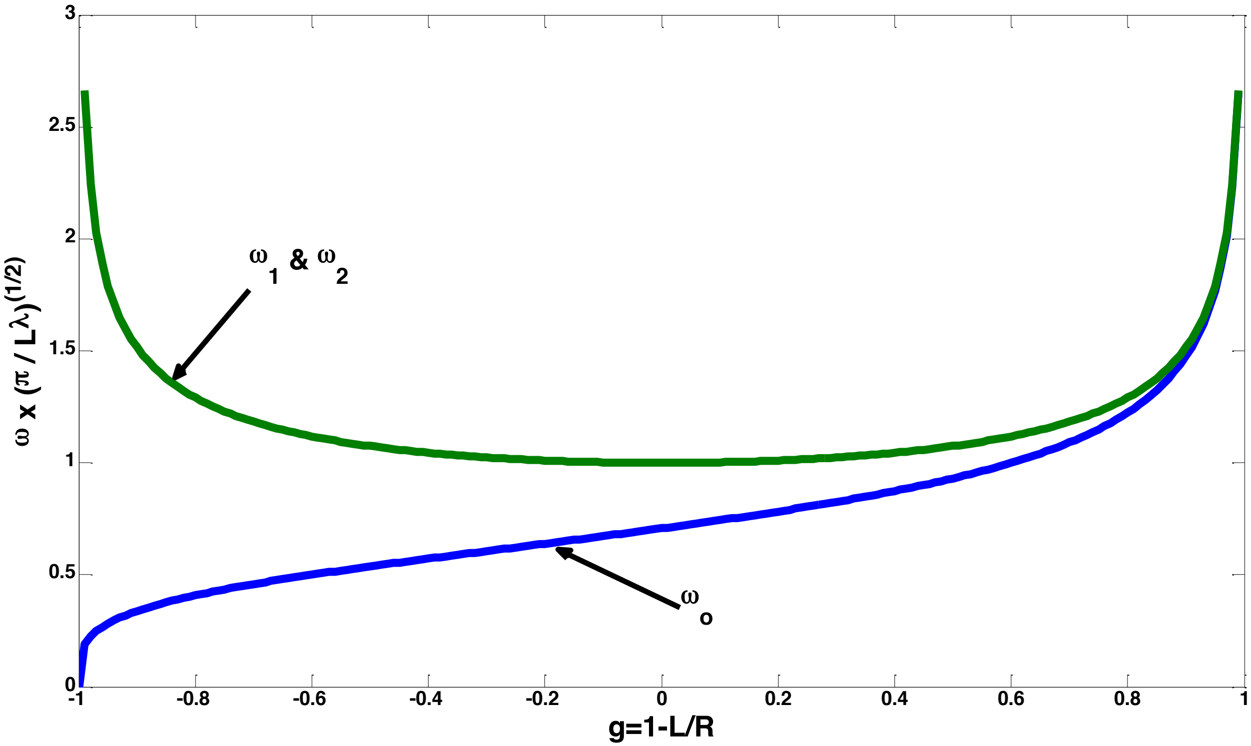
$$z_R^2 = L^2 \frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}$$

$$z_1 = -\frac{g_2 (1 - g_1)}{g_1 + g_2 - 2 g_1 g_2} L \qquad z_2 = z_1 + L$$

$$\omega_1^2 = \frac{L\lambda}{\pi} \sqrt{\frac{g_2}{g_1 (1 - g_1 g_2)}} \qquad \omega_2^2 = \frac{L\lambda}{\pi} \sqrt{\frac{g_1}{g_2 (1 - g_1 g_2)}}$$

Symmetric Resonators

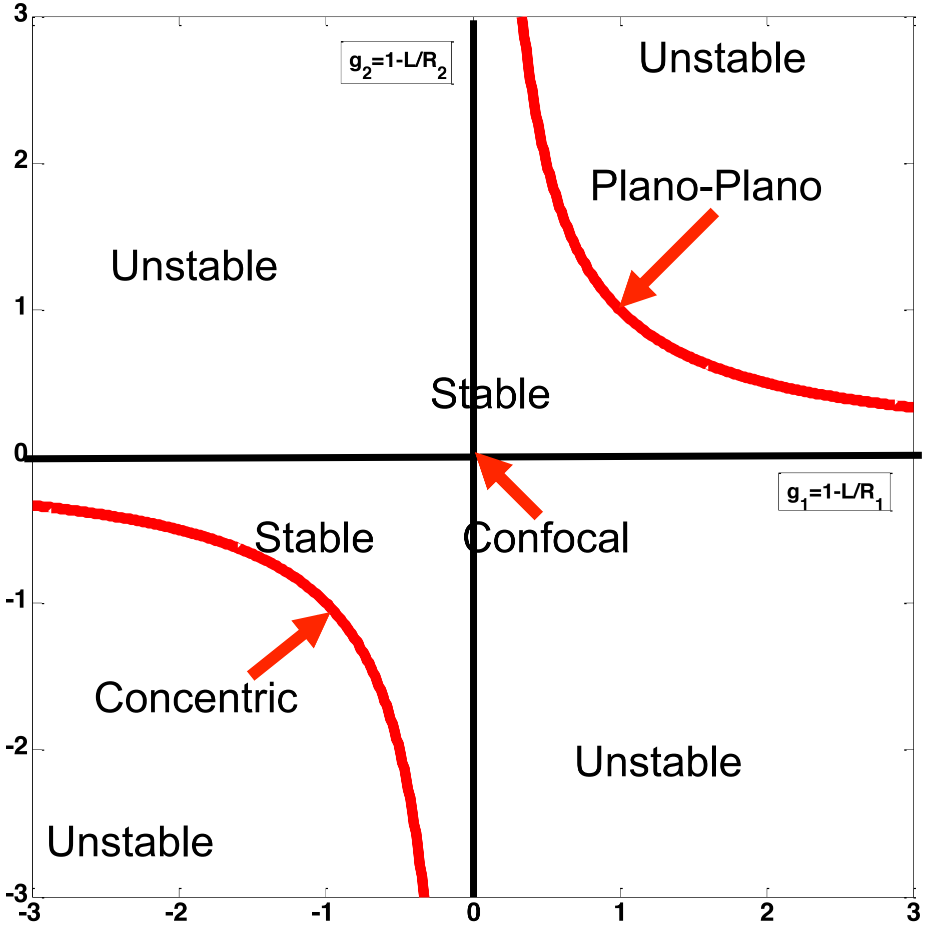
- $g_1 = g_2 = g$ $R_1 = R_2 = R$



$$\omega_o^2 = \frac{L\lambda}{\pi}\sqrt{\frac{1+g}{4(1-g)}}$$

$$\omega_1^2 = \omega_2^2 = \frac{L\lambda}{\pi}\sqrt{\frac{1}{1-g^2}}$$

Stable Cavity Designs



$$0 \le \left(1-\frac{L}{R_1}\right)\left(1-\frac{L}{R_2}\right) \le 1$$

Gaussian Beams

We derived three expressions:

Using these we can express the term $e^{-jp(z)}$ as $e^{-jp(z)} = \dfrac{1}{1 + z/q_0}$

$$\frac{1}{q_0} = -\frac{j\lambda}{\pi w_0^2} \qquad z_0 = \frac{\pi w_0^2}{\lambda}$$

We started by looking for a solution of the wave equation of the form

with a varying amplitude expressed as

$$e^{-jp(z)} = \frac{1}{1 - jz/z_0} = \frac{1}{\sqrt{1 + z^2/z_0^2}} e^{j\phi(z)}$$

where $\boxed{\phi(z) = \tan^{-1}(z/z_0)}$

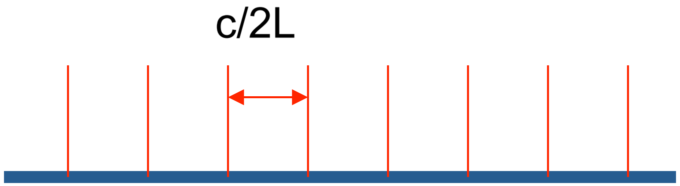
Now that we derived $q(z)$ and $p(z)$ we can write the full expression for the solution of the paraxial wave equation.

$$\mathcal{E}(\mathbf{r}) = \mathcal{E}_0(\mathbf{r}) e^{-jkz}$$

$$\mathcal{E}_0(\mathbf{r}) = A e^{-jk(x^2+y^2)/2q(z)} e^{-jp(z)}$$

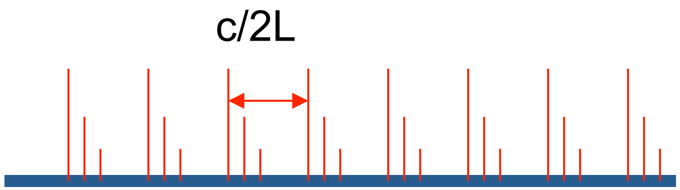
Frequency condition for resonant fields

•For a plano-plano cavity, the round trip condition of the phase needs to be a modulus 2π:



$$e^{i2k.L} = e^{i2\pi n} \qquad \upsilon = n\left(\frac{c_o}{2L}\right)$$

$$e^{i2k.L - i2(n+m+1)\left[\psi(z_1)-\psi(z_2)\right]} = e^{i2\pi q} \qquad \cos\left(\left[\psi(z_1)-\psi(z_2)\right]\right) = \sqrt{g_1 g_2}$$

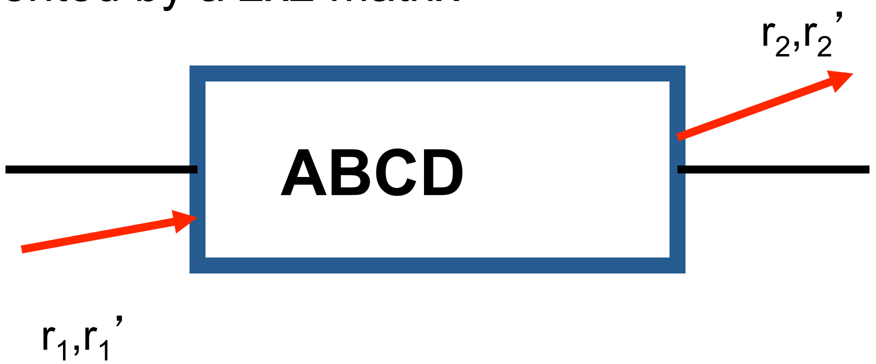


$$\upsilon = \left(\frac{c_o}{2L}\right)\left[q + (n+m+1)\frac{\cos^{-1}(\sqrt{g_1 g_2})}{\pi}\right]$$

Gaussian Beam Propagation ABCD Matrix

ABCD matrices formalism

•A linear optical system can be represented by a 2x2 matrix



$$r_2 = Ar_1 + Br_1'$$

$$r_2' = \frac{Dr_2 - r_1}{B}$$

•It can be shown that in the paraxial approximation for Gaussian beams the response of a linear system can be represented by the following expression

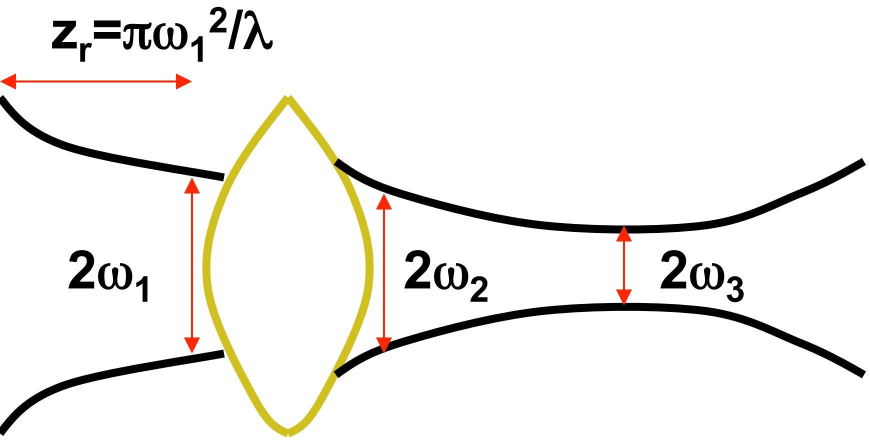
$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{j\lambda}{\pi w^2(z)}$$

Example of ABCD systems

$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$	**Translation**
$\begin{bmatrix} 1 & 0 \\ 0 & n1/n2 \end{bmatrix}$	**Refraction**
$\begin{bmatrix} 1 & 0 \\ (n_1 - n_2)/n_2 R & n_1 / n_2 \end{bmatrix}$	**Refractive Spherical surface**
$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$	**Thin Lens**
$\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$	**Reflective Spherical Mirror**
$\begin{bmatrix} \cos(L\sqrt{\frac{n_2}{n_o}}) & \frac{\sin(L\sqrt{\frac{n_2}{n_o}})}{\sqrt{n_o n_2}} \\ -\sqrt{n_o n_2}\sin(L\sqrt{\frac{n_2}{n_o}}) & \cos(L\sqrt{\frac{n_2}{n_o}}) \end{bmatrix}$	**Parabolic Duct**

Focusing a Gaussian beam



$$\frac{1}{q_1}=\frac{1}{R_1}-\frac{j\lambda}{\pi\omega_1^2}=-\frac{j\lambda}{\pi\omega_1^2}$$

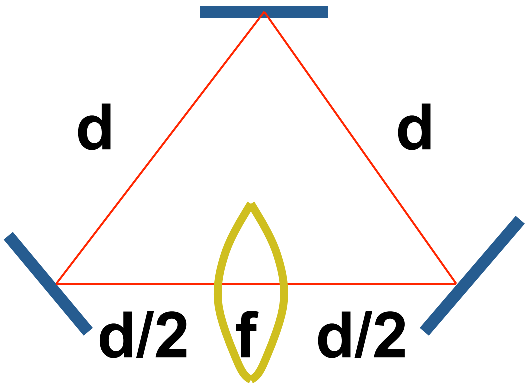
$$\frac{1}{q_2}=\frac{1}{q_1}-\frac{1}{f}\quad\Rightarrow q_2=\frac{-a+ib}{a^2+b^2}\ \text{with}\ a=\frac{1}{f}\quad b=\frac{\lambda}{\pi\omega_1^2}$$

$$q_3=q_2+Z\quad\Rightarrow\frac{1}{q_3}=\frac{1}{R_3}-\frac{j\lambda}{\pi\omega_3^2}$$

$$Z=\frac{a}{a^2+b^2}=\frac{f}{1+\left(f/z_R\right)^2}\quad\text{and}\quad\frac{\omega_3}{\omega_1}=\frac{f/z_R}{\sqrt{1+\left(f/z_R\right)^2}}$$

To achieve tight focusing f/zr needs to be small otherwise the focal spot is not at the focal plane of linear ray optics

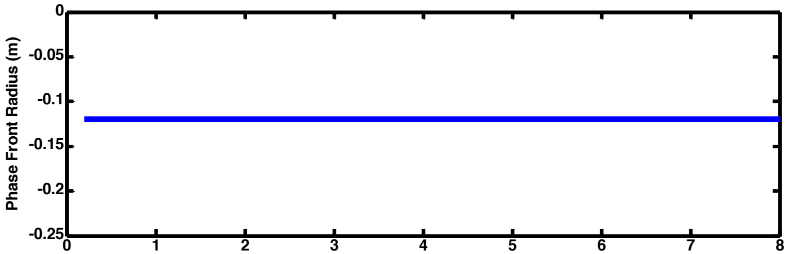
Example of Repetitive Structures



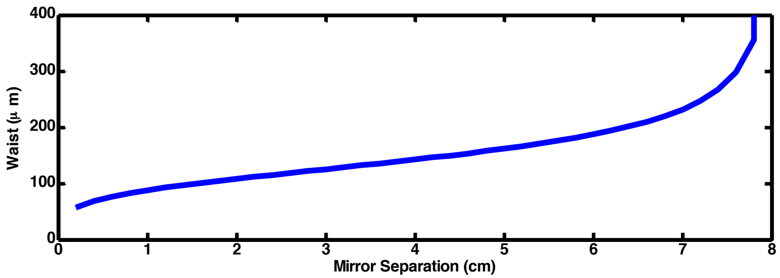
RingCavity.m

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & 3d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3d \\ -1/f & 1-3\,d/f \end{bmatrix}$$

$$\lambda^2 + (3\frac{d}{f} - 2)\lambda + 1 = 0$$



$$\lambda^{\pm} = 1 - \frac{3d}{2f} \pm \left(1 - \frac{4f}{3d}\right)^{1/2}$$



$$0 \le d/f \le 4/3$$ **Stability Condition**

Backup